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On the existence of plane curves with imposed multiple points

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Abstract

We prove that a plane curve of degree d with r points of multiplicity m must have

$$d \geq m(r-1)\prod_{i=2}^{r-1}\left(1-\frac{i}{i^2+r-1}\right), \qquad d > \left(\sqrt{r-1}-\frac{\pi}{8}\right)m.$$

MSC: Primary 14C20; secondary 14J26; 14M20; 14H20

1. Introduction

In [11] Nagata showed a counterexample to the 14th problem of Hilbert; in his construction, he proved that, for $n > 3$, a plane curve going with multiplicity at least m through n^2 points in general position must have degree strictly bigger than nm. Moreover, he conjectured that this result should also hold for a non-square number of points, that is, a curve with multiplicity m at $r \geq 10$ points in general position must have degree strictly bigger than $\sqrt{r}m$.

This conjecture has been proved only in some particular cases. In [4], Evain proves it for m small enough, concretely for $r > ([8m/(4m-1)](m+1))^2$. In the case of irreducible reduced curves, Xu proved in [12] the inequalities $d > \sqrt{r}m - 1/(2\sqrt{r-1})$ and $d > \sqrt{r-1}m$. As far as we know, the best bound known for the general case is what follows from Nagata's result, $d > [\sqrt{r}]m$, where $[\,\cdot\,]$ denotes the integral part.

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In this work we prove the inequalities

$$d \geq m(r-1)\prod_{i=2}^{r-1}\left(1-\frac{i}{i^2+r-1}\right), \qquad d > \left(\sqrt{r-1}-\frac{\pi}{8}\right)m$$

for all $r \geq 10$. This is better than the known bound for r in any interval $((n+\pi/8)^2+1, (n+1)^2)$, $n \in \mathbb{Z}$. Our approach is based on a specialization of the scheme consisting of r points in general position with multiplicity m to an appropriate cluster scheme supported at a single point.

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2. Definitions

Given an algebraic variety Z over an algebraically closed field k, and a closed subvariety Z' of Z, we will write $b : \mathrm{Bl}(Z, Z') \to Z$ for the blowing-up of Z with center Z'.

Let $p_1 \in S_0 = \mathbb{P}^2, p_2 \in S_1 = \mathrm{Bl}(S_0, \{p_1\}), \dots, p_r \in S_{r-1} = \mathrm{Bl}(S_{r-2}, \{p_{r-1}\})$. The set $\{p_1, p_2, \dots, p_r\}$ is called a *cluster* (see [2]) and the sequence $K = (p_1, p_2, \dots, p_r)$ is an *ordered cluster*. Here we will be concerned only with ordered clusters and we will call them simply clusters. Note that some of the points of a cluster can be identified to proper points of $\mathbb{P}^2$, whereas others may lie infinitely near to preceding points. A *system of multiplicities* for a cluster $K = (p_1, p_2, \dots, p_r)$ is a sequence of integers $(m) = (m_1, m_2, \dots, m_r)$, and a pair (K, m) where K is a cluster and (m) a system of multiplicities is called a *weighted cluster*. We review now briefly some known results on clusters; for the proofs, refer to [1,2], having in mind the minor change that we do not require all points in a cluster to be infinitely near to the first one.

Given a weighted cluster, we have an ideal sheaf and a zero-dimensional subscheme of $\mathbb{P}^2$ associated to it. Write $S_K = \mathrm{Bl}(S_{r-1}, \{p_r\})$ and denote by π_K the composition $S_K \to \mathbb{P}^2$ of the blowing-ups of the points of K. Let E_i be the pullback (total transform) in S_K of the exceptional divisor of blowing up p_i. Then the ideal sheaf

$$\mathscr{H}_{K,m} = (\pi_K)_* \mathcal{O}_{S_K}(-m_1E_1 - m_2E_2 - \cdots - m_rE_r)$$

defines a zero-dimensional subscheme of $\mathbb{P}^2$, and the stalks of $\mathscr{H}_{K,m}$ are complete ideals in the stalks of $\mathcal{O}_{\mathbb{P}^2}$. Conversely, if $\mathscr{I}$ is a coherent sheaf of ideals on $\mathbb{P}^2$ defining a zero-dimensional scheme whose stalks are complete ideals then there is a weighted cluster (K, m) such that $\mathscr{I} = \mathscr{H}_{K,m}$. We will call such schemes *cluster schemes*. Remark that a plane curve contains the cluster scheme defined by (K, m) if and only if it goes (virtually, as in [1,2]) through (K, m). This notion has already been considered by Greuel et al. in [5] (with the name *generalized singularity scheme*) and also by Harbourne in [8] (with the name *generalized fat point scheme*).

Given two points p_i, p_j in a cluster K with $j > i$, we say that p_j is *proximate* to p_i if and only if $j = i+1$ and p_j lies on the exceptional divisor $E \subset S_i$ of blowing up

p_i, or $j > i+1$ and p_j lies on the *strict* transform of E. The *proximity inequality* at p_i is

$$m_i \geq \sum_{p_j \text{ prox. to } p_i} m_j.$$

A cluster satisfying the proximity inequalities at all its points is called *consistent*. It happens that different weighted clusters $(K_1, m^{(1)})$ and $(K_2, m^{(2)})$ define the same cluster scheme. In this case $\mathscr{H}_{K_1,m^{(1)}} = \mathscr{H}_{K_2,m^{(2)}}$ and we will say that the two clusters are equivalent. For example, if p_2 is infinitely near p_1 then the weighted clusters

$$K_1 = (p_1), \quad m^{(1)} = (1),$$
$$K_2 = (p_1, p_2), \quad m^{(2)} = (0, 1),$$

are equivalent. However, if we ask that $m^{(i)} > 0$ for all i and (K,m) be consistent, then the cluster scheme determines the weighted cluster, but for the ordering of points.

Given an arbitrary weighted cluster (K,m) there is a procedure called *unloading* (see [2,4,3, IV.II] or [1]) which gives a new system of multiplicities (m') such that (K, m') is consistent and equivalent to (K,m). In each step of the procedure, one *unloads* some amount of multiplicity on a point p_i whose proximity inequality is not satisfied, from the points proximate to it. This means that there is an integer $n > 0$ such that, increasing the multiplicity of p_i by n and decreasing the multiplicity of every point proximate to p_i by n, the resulting weighted cluster is equivalent to (K,m) and satisfies the proximity inequality at p_i. In other words, if $\tilde{E}_i \subset S_K$ is the strict transform of the exceptional divisor of blowing-up p_i, $D = -m_1E_1 - m_2E_2 - \cdots - m_rE_r$ and $\tilde{E}_i \cdot D < 0$ then one chooses n as the minimal integer with $\tilde{E}_i \cdot (D - n\tilde{E}_i) \geq 0$ and replaces D by $D - n\tilde{E}_i$. A finite number of unloading steps lead to the desired equivalent consistent cluster.

Let T be a variety, which for the moment we will think of as a fixed base for our constructions. Let $p : X \to T$ be a smooth morphism of relative dimension n, and let $i : Y \to X$ be a smooth embedding over p.

Let us consider the diagonal morphism $\Delta := Id_Y \times_T i$ which makes the following diagram commutative:

$$\begin{tikzcd}
Y \arrow[dr, "\Delta"] \arrow[drr, "i"] \arrow[ddr, "Id_Y"'] & & \\
& Y \times_T X \arrow[r, "p_X"'] \arrow[d, "p_Y"] & X \arrow[d, "p"] \\
& Y \arrow[r, "p \circ i"'] & T
\end{tikzcd}$$

The image $\Delta(Y)$ is a closed smooth subvariety isomorphic to Y. Consider the blowing-up

$$\mathrm{BF}(X, Y, T) := \mathrm{Bl}(Y \times_T X, \Delta(Y)) \xrightarrow{b} Y \times_T X,$$

and the commutative diagram

$$\begin{tikzcd} BF\,(X,\,Y,\,T) \arrow[r, "p_x \circ b"] \arrow[d, "p_y \circ b"'] & X \arrow[d] \\ Y \arrow[r] & T \end{tikzcd}$$

We call $\pi = p_X \circ b$ and $q = p_Y \circ b$. As Δ is a smooth embedding over p, it follows that q is smooth, of relative dimension n (see [6, 19.4]). We call

$$\mathrm{BF}(X,Y,T) \xrightarrow{q} Y,$$

the family of blowing up X at the points of Y. We are going to see that the morphism $\mathrm{BF}(X,Y,T) \xrightarrow{\pi} X$, makes the fibers of q into ordinary blowing-ups, hence the name. Given $y \in Y$, with $p(y) = t$, call $\mathrm{BF}(X,Y,T)_y = \mathrm{BF}(X,\,Y,\,T) \times_Y \{y\}$ and $X_t = X \times_T \{t\}$. Note that $y \in X_t$.

Proposition 2.1. *For every point $y \in Y$, and $t = p(y) \in T$ consider the blowing-up $b : \mathrm{Bl}(X_t, \{y\}) \to X_t$. Then there is a unique isomorphism*

$$\mathrm{Bl}(X_t, \{y\}) \xrightarrow{\psi} \mathrm{BF}(X,Y,T)_y,$$

satisfying $b = \pi|_{\mathrm{BF}(X,\,Y,\,T)_y} \circ \psi$.

Proof. Follows from [10, 2.4], as $\Delta(Y)$ is obviously a local complete intersection, flat over Y. $\square$

3. Varieties of clusters

Take now $X_{-1} = \operatorname{Spec} k$, $X_0 = \mathbb{P}^2_k$; $p_0 : \mathbb{P}^2_k \to \operatorname{Spec} k$, and define recursively X_i, p_i as the blowing-up family

$$X_i = \mathrm{BF}(X_{i-1}, X_{i-1}, X_{i-2}) \xrightarrow{p_i} X_{i-1}.$$

The morphisms p_i are in this case projective and smooth of relative dimension 2, so their fibers are projective smooth surfaces. We have also morphisms $\pi_i : X_i \to X_{i-1}$ whose restrictions to the fibers of p_i are, by Proposition 2.1, the blowing-ups of the points of the fibers of p_{i-1}. To simplify notations, let us say $\pi_{r,i} = \pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_r$, $p_{r,i} = p_i \circ p_{i+1} \circ \cdots \circ p_r$. If there is no confusion possible on r, we will also write p_i for $p_{r,i}$, so $p_i(x)$ is a point in X_{i-1}, defined for all x in X_r, $r \geq i$. For any point $x \in X_i$, we will call $S_x = (X_i)_{p_i(x)} = X_i \times_{X_{i-1}} \{p_i(x)\}$ the surface containing x. Recall that for any cluster K, $\pi_K : S_K \to \mathbb{P}^2$ is the composition of the blowing-ups of the points in K.

The following proposition makes the set of all clusters with r points into an algebraic variety.

Proposition 3.1. *For every $r \geq 1$ there is a bijection*

$$X_{r-1} \xrightarrow{K} \{\text{clusters of } r \text{ points}\}$$

and, for every $x \in X_{r-1}$, a unique isomorphism $\psi_x : S_{K(x)} \to (X_r)_x$ such that $\pi_K = \pi_{r,0|(X_r)_x} \circ \psi_x$.

Proof. Follows from [7, 1.2], since there is an obvious bijection

$$\{\text{ordered blowing-ups at } r \text{ points}\} \to \{\text{ordered clusters of } r \text{ points}\}$$
$$S_K \mapsto K. \qquad \square$$

Notice that the ordering of points in clusters is essential in Proposition 3.1. If two clusters differing only in the order of points were considered equal, as in [2], then injectivity would fail. From now on identify the set of clusters of r points to the variety X_{r-1}.

For every pair of integers $1 \leq i < j \leq r$ there is a subset of X_{r-1} containing exactly those clusters $K = (x_1, x_2, \dots, x_r)$ for which x_j is proximate to x_i. It can be proved that these subsets are constructible subsets of X_{r-1}; we will focus on some of them which are irreducible closed varieties.

Call F_i the exceptional divisor of

$$X_i \xrightarrow{b_i} X_{i-1} \times_{X_{i-2}} X_{i-1}.$$

Because of Proposition 2.1 the pullback of F_i to $(X_i)_{p_i}$, is the exceptional divisor E_i of blowing up p_i in S_{p_i}. It is clear that $p_i(K)$ is proximate to $p_{i-1}(K)$ if and only if $p_i(K) \in F_{i-1}$. So there is a closed subvariety

$$Y_{r-1} := \bigcap_{i=2}^{r} p_i^{-1}(F_{i-1}) \subset X_{r-1},$$

containing exactly those clusters K for which $p_{i+1}(K)$ is proximate to $p_i(K)$ for all i. It is also clear that $p_{r-1}(Y_{r-1}) = Y_{r-2}$, if we allow $Y_0 = \mathbb{P}^2$.

Lemma 3.2. *For all r, there is a closed immersion*

$$\mathrm{BF}(X_{r-1}, Y_{r-1}, X_{r-2}) \xrightarrow{i} X_r$$

such that Y_r is the image of the exceptional divisor F_r' of

$$\mathrm{BF}(X_{r-1}, Y_{r-1}, X_{r-2}) \xrightarrow{b} X_{r-1} \times_{X_{r-2}} Y_{r-1}.$$

Proof. The closed immersion i is the strict transform of the closed immersion

$$Y_{r-1} \times_{X_{r-2}} X_{r-1} \to X_{r-1} \times_{X_{r-2}} X_{r-1}$$

(see [9, II,7.15]). By definition of the Y_r we know that $Y_r = F_r \cap p_r^{-1}(Y_{r-1})$, and obviously $i(F_r') \subset F_r$, and

$$(p_r \circ i)(\mathrm{BF}(X_{r-1}, Y_{r-1}, X_{r-2})) = Y_{r-1},$$

so $i(F_r') \subset Y_r$. On the other hand, if $y_r \in Y_r$ then $p_r(y_r) = y_{r-1} \in Y_{r-1}$, so

$$y_r \in S_{y_r} \cong \mathrm{Bl}(S_{y_{r-1}}, \{y_{r-1}\}) \cong \mathrm{BF}(X_{r-1}, Y_{r-1}, X_{r-2})_{y_{r-1}},$$

which implies $y_r \in i(F_r')$. So $Y_r \subset i(F_r')$, and the proof is complete. □

Corollary 3.3. *For all r, Y_r together with the restricted morphism $p_r : Y_r \to Y_{r-1}$ is a $\mathbb{P}^1$-bundle, and Y_r is irreducible.*

To deal with the proximity relations between points p_i and p_j where $j > i+1$ we need some control on the strict transforms of the exceptional divisor of blowing up p_i. In contrast to what we have seen in the case $j = i+1$, there is no variety $\tilde{F}_i \subset X_{j-1}$ whose pullback to $(X_{j-1})_{p_{j-1}(K)}$ is the desired strict transform for all K. To overcome this difficulty we restrict ourselves to clusters in Y_{r-1} and define varieties $D_{i,j} \subset X_{j-1}$ whose pullback to $(X_{j-1})_{p_{j-1}(K)}$ is the strict transform of the exceptional divisor of blowing up $p_i(K)$ if $p_{j-1}(K)$ is proximate to $p_i(K)$ and empty in any other case. Let first

$$D_{i,i+1}' = D_{i,i+1} = Y_i.$$

Suppose now we have defined $D_{i,j-1} \subset X_{j-2}$ and $D_{i,j-1}' = D_{i,j-1} \cap Y_{j-2}$, such that the morphism $p_{j-2}|_{D_{i,j-1}}$ is smooth of relative dimension 1 (observe that for $D_{i,i+1} = Y_i$ this is so). As there is a closed immersion $D_{i,j-1} \times_{X_{j-3}} D_{i,j-1}' \to X_{j-2} \times_{X_{j-3}} X_{j-2}$ there is also a closed immersion (its strict transform)

$$D_{i,j} = \mathrm{BF}(D_{i,j-1}, D_{i,j-1}', X_{j-3}) \xrightarrow{i} X_{j-1},$$

which we take as the definition of $D_{i,j}$. Moreover as $p_{j-2}|_{D_{i,j-1}}$ is smooth of relative dimension $1, \Delta(D_{i,j-1})$ has codimension 1 in $D_{i,j-1} \times_{X_{j-3}} D_{i,j-1}$ and

$$\mathrm{BF}(D_{i,j-1}, D_{i,j-1}', X_{j-3}) \xrightarrow{b} D_{i,j-1} \times_{X_{j-3}} D_{i,j-1}'$$

is an isomorphism. We have

$$D_{i,j}' = D_{i,j} \cap Y_{j-1} \cong \Delta(D_{i,j-1}') \subset \mathrm{BF}(D_{i,j-1}, D_{i,j-1}', X_{j-3}).$$

So $D_{i,j}'$ is isomorphic to $D_{i,j-1}'$, and $p_{j-1}|_{D_{i,j}}$ is smooth of relative dimension 1.

We will call (i,j)-proximity variety the subvariety $P_{i,j} = p_j^{-1}(D_{i,j}') \subset Y_{r-1}$.

Lemma 3.4. *In a cluster $K \in Y_{r-1}$ the points $p_{i+1}, p_{i+2}, \ldots, p_j$ are proximate to p_i if and only if K lies in the (i,j)-proximity variety. Furthermore, the proximity varieties are irreducible and there are inclusions*

$$P_{i,i+1} \supset P_{i,i+2} \supset \cdots \supset P_{i,r}.$$

Proof. The first part will clearly be proved if we show that

$$D_{i,j} \times_{X_{j-1}} \{p_{j-1}\} \subset S_{p_j}$$

is the strict transform of E_i. This comes out easily by induction on $j-i$. For $j-i=1$, it is immediate by Proposition 2.1. For $j-i>1$, Proposition 2.1 gives

$$D_{i,j} \times_{X_{j-1}} \{p_{j-1}\} = \mathrm{Bl}(D_{i,j-1} \times_{X_{j-2}} \{p_{j-2}\}, p_{j-1}),$$

that is, the strict transform in S_{p_j} of $D_{i,j-1} \times_{X_{j-2}} \{p_{j-2}\}$, which by the induction hypothesis is the strict transform of E_i in $S_{p_{j-1}}$, so we are done.

From their own definition, the $D'_{i,j}$ are all isomorphic to Y_i, which is irreducible. Induction on $r-j$ gives the irreducibility of the $P_{i,j}$. Indeed, if $P_{i,j}^{(r-1)} = (p_j|_{Y_{r-2}})^{-1}(D_{i,j})$ is irreducible then its preimage by $p_{r-1}|Y_{r-1}$ must be irreducible also, because $Y_{r-1} \to Y_{r-2}$ is a projective space bundle.

The inclusions between the $P_{i,j}$ are clear, from the first part of the lemma. □

Lemma 3.4 shows that there are subsets $U_{1,i}$ open and dense in $P_{1,i}$ which contain all those clusters K with
- $p_j(K)$ proximate to $p_{j-1}(K)$, $j = 2, \ldots, r$,
- $p_j(K)$ proximate to $p_1(K)$, $2 \le j \le i$

and no other proximity relations.

Lemma 3.5. *Let $(m) = (m_1, m_2, \ldots, m_r)$ be a system of multiplicities, and call $M = \sum_{j=2}^r m_j$. Define $\alpha_i = (i-1)/(r-1)$ and $\beta_i = 1 - (i-1)/((i-1)^2 + r - 1)$. Suppose that for some $i \in \{2, 3, \ldots, r\}$ and $A \in \mathbb{R}$ the inequalities*

$$\frac{(i-2)m_1 + M}{(i-2)\alpha_{i-1} + 1} \ge A, \quad m_1 \ge \alpha_{i-1} A,$$

are satisfied. Then there is a system of multiplicities (m') which is equivalent to (m) for all clusters in $U_{1,i}$ and satisfies

$$\frac{(i-1)m'_1 + M'}{(i-1)\alpha_i + 1} \ge \beta_i A, \tag{1}$$

$$m'_1 \ge \alpha_i \beta_i A. \tag{2}$$

Proof. We know that for a given cluster of r points K there is a system of multiplicities (m'), consistent and equivalent to (m), which is obtained from (m) by the unloading procedure. The unloading procedure depends only on the multiplicities and the proximity relations, and so it is the same for all clusters in $U_{1,i}$.

Due to the proximity relations which hold for the points of a cluster in $U_{1,i}$, when an unloading step is applied to the point p_j, $1 < j < r$ the only point whose multiplicity is decreased is p_{j+1}, so m_1 and M remain unchanged. When an unloading step is applied to p_1, the points whose multiplicity is decreased are $\{p_2, p_3, \ldots, p_i\}$, so if m_1 is increased by n, M is decreased by $(i-1)n$. In both cases, the quantity $(i-1)m_1 + M$ remains the same. When an unloading step is applied to p_r, which happens only when its multiplicity has become negative, then one replaces it by zero, so $(i-1)m_1 + M$ might increase, but does never decrease. After the complete unloading procedure we get

$$\begin{aligned}(i-1)m'_1 + M' &\ge (i-1)m_1 + M = (i-2)m_1 + M + m_1 \\ &\ge ((i-2)\alpha_{i-1} + 1)A + \alpha_{i-1}A = ((i-1)\alpha_i + 1)\beta_i A.\end{aligned}$$

This proves (1). To see (2), we multiply this inequality by α_i, so we get

$$\alpha_i((i-1)m'_1 + M') \geq ((i-1)\alpha_i + 1)\alpha_i\beta_i A.$$

On the other hand, as (m') is consistent, (K,m) must satisfy all the proximity inequalities, and these imply easily

$$m'_1 - \alpha_i M' \geq 0.$$

If we add both inequalities, we obtain (2). □

4. The bound

Let $F_i^{(r)}$ be the pullback of F_i by $\pi_{r,i} : X_r \to X_i$. Let $[F_0]^{(r)}$ be the pullback to X_r by $\pi_{r,0}$ of the class of a line in $\mathbb{P}^2$. For any cluster $K \in X_{r-1}$ and $i > 0$, the pullback to the surface S_K of $F_i^{(r)}$ by the inclusion is obviously the same as the pullback E_i of the class of the exceptional divisor of blowing up p_i in $S_{p_i(K)}$ by $\pi_{r,i}|_{S_K}$. Similarly, the pullback of $[F_0]^{(r)}$ to S_K is the same as the pullback $[E_0]$ of the class of a line by $\pi_{r,0}|_{S_K}$. All together, we have

$$\mathcal{O}_{X_r}(F_i^{(r)}) \otimes_{X_{r-1}} k(K) = \mathcal{O}_{S_K}(E_i) \tag{3}$$

for all i. Given an integer d we define

$$\mathcal{J}_{d,m} = \mathcal{O}_{X_r}(dF_0^{(r)} - m_1F_1^{(r)} - m_2F_2^{(r)} - \cdots - m_rF_r^{(r)}).$$

Equality (3) and the projection formula show that, for every cluster $K \in X_{r-1}$,

$$\mathcal{H}_{K,m}(d) = \mathcal{H}_{K,m} \otimes \mathcal{O}_{\mathbb{P}^2}(d) = (\pi_K)_*(\mathcal{J}_{d,m} \otimes_{X_{r-1}} k(K))$$

and $H^0(\mathcal{H}_{K,m}(d)) \cong H^0(\mathcal{J}_{d,m} \otimes_{X_{r-1}} k(K))$.

In our specialization, we start from a cluster K consisting of r points in general position, to specialize it, step by step, to the closed subvarieties $P_{1,i}$. We obtain the following theorem:

Theorem 4.1. *If a plane curve of degree d passes with multiplicity m through r points in general position, then*

$$d \geq m(r-1)\prod_{i=2}^{r-1}\left(1 - \frac{i}{i^2 + r - 1}\right). \tag{4}$$

Proof. Let $\mathcal{J}$ and $\mathcal{H}$ be the sheaves defined above. We start from the system of multiplicities $(m) = (m,m,\ldots,m)$. We have to prove that for general $K \in X_{r-1}$, the inequality

$$H^0((\mathcal{J}_{d,m}) \otimes_{X_{r-1}} k(K)) \cong H^0(\mathcal{H}_{K,m}(d)) \neq 0$$

implies (4), so assume this inequality holds for general K. As $X_r \to X_{r-1}$ is smooth, the invertible sheaf $\mathcal{J}_{d,m}$ is flat over X_{r-1}, so by the semicontinuity theorem [9, III,12.8] we have

$$H^0(\mathscr{H}_{K,m}(d)) \neq 0$$

for all $K \in P_{1,i}$ and any i.

Now for $K \in P_{1,3}$ the system of multiplicities (m) is not consistent. We can find by unloading multiplicities a consistent system $(m^{(3)})$ which is equivalent to (m) for all clusters in $U_{1,3}$. Applying Lemma 3.5 with $(m)=(m,m,\dots,m)$, $M=(r-1)m$ and $i=3$, we have

$$\frac{(i-2)m_1+M}{(i-2)\alpha_{i-1}+1} = \frac{m+(r-1)m}{\alpha_2+1} = m(r-1),$$

so we can take $A = A_2 = m(r-1)$ and the lemma gives

$$\frac{2m_1^{(3)}+M^{(3)}}{2\alpha_3+1} \geq m(r-1)\beta_3, \quad m_1^{(3)} \geq m(r-1)\alpha_3\beta_3.$$

As $(m^{(3)})$ is equivalent to (m) for all clusters in $U_{1,3}$, we have

$$H^0(\mathscr{H}_{K,m^{(3)}}(d)) = H^0(\mathscr{H}_{K,m}(d)) \neq 0$$

if $K \in U_{1,3}$. As $U_{1,3}$ is open and dense in $P_{1,3}$, and $P_{1,4} \subset P_{1,3}$, the semicontinuity theorem applied to the new sheaf $\mathscr{I}_{d,m^{(3)}}$ implies

$$H^0(\mathscr{H}_{K,m^{(3)}}(d)) \neq 0$$

for all $K \in P_{1,4}$. The new system of multiplicities need not be (but in fact could be) consistent for $K \in P_{1,4}$. In any case we can find a new system $(m^{(4)})$ (which could be equal to $(m^{(3)})$) to use here. We apply Lemma 3.5 to the new situation, with $A_3 = m(r-1)\beta_3$, and we obtain

$$\frac{3m_1^{(4)}+M^{(4)}}{3\alpha_4+1} \geq m(r-1)\beta_3\beta_4, \quad m_1^{(4)} \geq m(r-1)\alpha_4\beta_3\beta_4.$$

Iterating the process we finally get a system $(m^{(r)}) = (m_1^{(r)}, m_2^{(r)}, \dots, m_r^{(r)})$, with

$$m_1^{(r)} \geq m(r-1)\alpha_r \prod_{i=3}^{r} \beta_i = m(r-1) \prod_{i=2}^{r-1} \left(1 - \frac{i}{i^2+r-1}\right)$$

and

$$H^0(\mathscr{H}_{K,m^{(r)}}(d)) \neq 0$$

for all $K \in P_{1,r}$. It is clear that this implies $d \geq m_1^{(r)}$. $\square$

The reader may note that the proof of Theorem 4.1 is valid for any divisor class on an irreducible smooth projective surface S, except for the last step, namely $d \geq m_1^{(r)}$, which assumes $C \subset S = \mathbb{P}^2$. The specialization of a set of multiple points to a cluster scheme containing a point of multiplicity $m' \geq m(r-1)\prod_{i=2}^{r-1}(1 - i/(i^2+r-1))$ holds thus on any such surface.

5. A calculation

The aim of this section is to compare the bound of Theorem 4.1 with Nagata's conjecture (which reads $d > m\sqrt{r}$), and with previously known results. We obtain the following:

Proposition 5.1. *Let $n \geq 9$ be a natural number. Then*

$$n\prod_{i=2}^{n}\left(1-\frac{i}{i^2+n}\right) > \sqrt{n}-\frac{\pi}{8}.$$

This has an immediate corollary:

Corollary 5.2. *If a plane curve of degree d passes with multiplicity m through $r \geq 10$ points in general position, then*

$$d > m\left(\sqrt{r-1}-\frac{\pi}{8}\right).$$

Proof of Proposition 5.1. The goal is to bound

$$b = n\prod_{i=2}^{n}\left(1-\frac{i}{i^2+n}\right) = n\prod_{i=1}^{n-1}\left(1-\frac{i}{i^2+n}\right) = n\prod_{i=1}^{n-1}\frac{n+i^2-i}{i^2+n}$$

below. This can be rewritten as

$$n\frac{(1/n)\prod_{i=2}^{n}(n+i^2-i)}{\prod_{i=1}^{n-1}(i^2+n)} = \prod_{i=1}^{n-1}\frac{n+(i+1)^2-(i+1)}{i^2+n} = \prod_{i=1}^{n-1}\left(1+\frac{i}{i^2+n}\right).$$

We thus have

$$b^2 = n\prod_{i=1}^{n-1}\left(1-\frac{i}{i^2+n}\right)\prod_{i=1}^{n-1}\left(1+\frac{i}{i^2+n}\right) = n\prod_{i=1}^{n-1}\left(1-\left(\frac{i}{i^2+n}\right)^2\right).$$

Let $1-\varepsilon = \prod_{i=1}^{n-1}(1-(i/(i^2+n))^2)$ and let $1+\delta = \prod_{i=1}^{n-1}(1+(i/(i^2+n))^2)$. Then $\varepsilon = 1-\prod_{i=1}^{n-1}(1-(i/(i^2+n))^2)$ and $\delta = -1+\prod_{i=1}^{n-1}(1+(i/(i^2+n))^2)$ both involve the same terms, except that they occur with signs in ε, so $0<\varepsilon<\delta$. Thus $1-\delta < 1-\varepsilon$, and so $b^2 > n(1-\delta)$.

We can bound b^2 (and hence b) below by bounding $1+\delta$ (and hence δ) above. But $\log(1+x) \leq x$ so $\log\prod_{i=1}^{n-1}(1+(i/(i^2+n))^2) \leq \sum_{i=1}^{n-1}(i/(i^2+n))^2$.

The Fourier series for $\sinh\sqrt{n}x$ on $[-\pi,\pi]$ is

$$\frac{2}{\pi}(\sinh\sqrt{n}\pi)\sum_{i\geq 1}(-1)^i\frac{-i}{i^2+n}\sin ix$$

so Parseval's identity gives

$$\left(\frac{\pi}{2\sinh\sqrt{n}\pi}\right)^2\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh^2\sqrt{n}x\,\mathrm{d}x = \sum_{i\geq 1}\left(\frac{i}{i^2+n}\right)^2.$$

The integral can be exactly evaluated; we get

$$\int_{-\pi}^{\pi} \sinh^2 \sqrt{n}x \,\mathrm{d}x = -\pi + \frac{1}{2\sqrt{n}} \sinh 2\sqrt{n}\pi.$$

Also,

$$\sum_{i \geq n} \left(\frac{i}{i^2+n}\right)^2 \geq \sum_{i \geq n} \left(\frac{i}{i^2+i}\right)^2 \geq \int_n^{\infty} \left(\frac{1}{x+1}\right)^2 \mathrm{d}x = \frac{1}{n+1}.$$

Thus we have

$$\begin{aligned}\sum_{i=1}^{n-1} \left(\frac{i}{i^2+n}\right)^2 &\leq \left(\frac{\pi}{2\sinh\sqrt{n}\pi}\right)^2 \frac{1}{\pi} \left(-\pi + \frac{1}{2\sqrt{n}} \sinh 2\sqrt{n}\pi\right) - \frac{1}{n+1} \\ &\leq \frac{\pi}{8\sqrt{n}} \frac{\sinh 2\sqrt{n}\pi}{\sinh^2 \sqrt{n}\pi} - \frac{1}{n+1}.\end{aligned}$$

Define $t = \mathrm{e}^{\sqrt{n}\pi}$, so

$$\begin{aligned}\frac{\sinh 2\sqrt{n}\pi}{2\sinh^2\sqrt{n}\pi} = \frac{t+1/t}{t-1/t} &= \left(1+\frac{1}{t^2}\right)\left(1+\frac{1}{t^2}+\frac{1}{t^4}+\cdots\right) \\ &= \left(1+\frac{1}{t^2}\right)\left(1+\frac{1}{t^2}\left(1+\frac{1}{t^2}+\cdots\right)\right) \leq \left(1+\frac{1}{t^2}\right)\left(1+\frac{1.5}{t^2}\right) \\ &\leq \left(1+\frac{3}{t^2}\right),\end{aligned}$$

because $n \geq 9$, and

$$\sum_{i=1}^{n-1} \left(\frac{i}{i^2+n}\right)^2 \leq \frac{\pi}{4\sqrt{n}} + \frac{1}{t^2} - \frac{1}{n+1}.$$

But $\mathrm{e}^{\sqrt{n}\pi} \geq 3n$ (look at the tangent line to $\mathrm{e}^{\sqrt{n}\pi}$ at n=9), so $\mathrm{e}^{2\sqrt{n}\pi} \geq 3n^2 \geq (n+2)(n+1)$ hence $1/t^2 \leq 1/((n+2)(n+1))$; therefore

$$\sum_{i=1}^{n-1} \left(\frac{i}{i^2+n}\right)^2 \leq \frac{\pi}{4\sqrt{n}} - \frac{1}{n+2}.$$

This means $\delta \leq -1 + \mathrm{e}^{\pi/(4\sqrt{n})-1/(n+2)}$, hence

$$\begin{aligned}b^2 &\geq n(1-\delta) \geq n(2-\mathrm{e}^{\pi/(4\sqrt{n})-1/(n+2)}) \\ &= n\left(1-\frac{\pi}{4\sqrt{n}}+\frac{1}{n+2}-\frac{1}{2!}\left(\frac{\pi}{4\sqrt{n}}-\frac{1}{n+2}\right)^2-\cdots\right) \\ &\geq n\left(1-\frac{\pi}{4\sqrt{n}}+\frac{1}{n+2}-\frac{1}{2!}\left(\frac{\pi}{4\sqrt{n}}\right)^2-\cdots\right)\end{aligned}$$

and by comparison with a geometric series, this last is at least as big as

$$n - \frac{\pi\sqrt{n}}{4} + \frac{n}{n+2} - n\frac{2u^2}{1-u},$$

where $u=(\pi/(4\sqrt{n}))/2 \leq \pi/24$, so $1/(1-u) \leq 1/(1-(\pi/24)) \leq 1.2$, so $-n2u^2/(1-u) \geq -2.4nu^2 \geq -0.4$; i.e., $b^2 \geq n - \pi\sqrt{n}/4 + n/(n+2) - 0.4 \geq n - \pi\sqrt{n}/4 + 9/11 - 0.4$. Finally, $(\sqrt{n} - \pi/8)^2 = n - \pi\sqrt{n}/4 + \pi^2/64$, and $9/11 - 0.4 > \pi^2/64$, so $b \geq \sqrt{n} - \pi/8$, as required. $\square$

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